CH 4: Deflection and Stiffness

Stress analyses are done to ensure that machine elements will not fail due to stress levels exceeding the allowable values. However, since we are dealing with deformable bodies (not rigid), deflections should be considered also where they are <u>in many cases more limiting</u> than stresses. Take for example shafts where excessive deflection will interfere with the function of the elements mounted on the shaft and might cause failure of the system, thus usually shafts are designed based on deflections rather than stresses.

Spring Rates

In <u>most types</u> of loading situations, the stress developed in the element (bar, shaft, beam, etc.) is <u>linearly</u> related to the loading. As long as the stress in the material remains <u>within the linear elastic</u> region, the stress is also linearly related to the deflection. Therefore, there is a linear relation between load and deflection and elements under loading behave similar to <u>linear springs</u>, and thus we can define the spring rate or spring constant for the element as:

$$k = \frac{Loading}{Deflection}$$

Axial, lateral, moment, torque, etc.

Axial, lateral, bending, twisting, etc.

Tension, Compression and Torsion

- For a <u>bar</u> with constant cross-section the deformation is found as:

$$\delta = \frac{F\,L}{A\,E}$$

  Thus, the spring constant for an axially loaded bar is:

$$k = \frac{A\,E}{L}$$

- For a <u>round shaft</u> subjected to torque, the angular deflection is found as:

$$\theta = \frac{TL}{GJ} \qquad \theta \text{ in radians}$$

Class Notes by: Dr. Ala Hijazi

Thus, the spring constant is:

$$k = \frac{G\,J}{L}$$

Deflection Due to Bending

The deflection of beams is much larger than that of axially loaded elements, and thus the problem of bending is <u>more critical</u> in design than other types of deformation.

- Shafts are treated as beams when analyzed for lateral deflection.
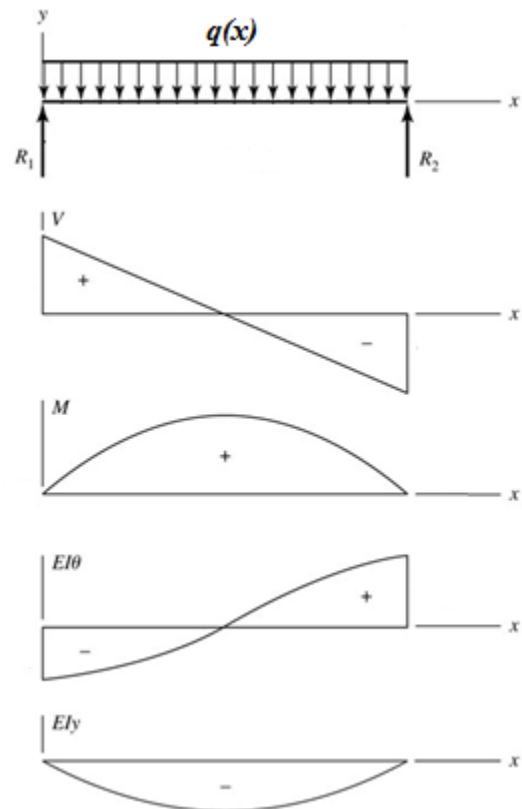
- The beam governing equations are:

Load intensity $\dfrac{q}{E\,I} = \dfrac{d^4y}{dx^4}$

Shear force $\dfrac{V}{EI} = \dfrac{d^3y}{dx^3}$

*Moment* $\dfrac{M}{E\,I} = \dfrac{d^2y}{dx^2}$

Slope $\theta = \dfrac{dy}{dx}$

Deflection $y = f(x)$

- Knowing the load intensity function, we can integrate four times (*using the known boundary conditions to evaluate the integration constants*) to get the deflection.
- However, in most cases, the <u>expression of bending moment</u> is easy to find (*using sections*) and thus we start from the moment governing equation and integrate to get the slope and deflection.

Example: The beam shown has constant cross-section and it is made from homogeneous isotropic material. Find the deflection, slope and the location and value of maximum slope.
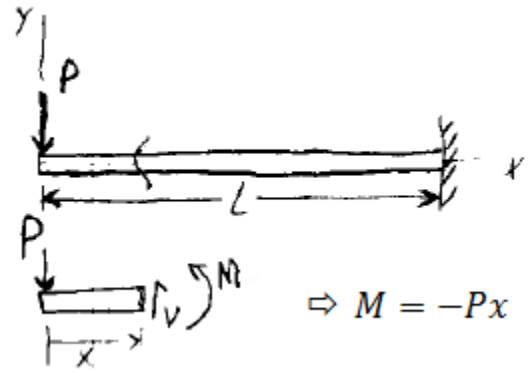
Solution:

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{Px}{EI}$$

Integrating → $\frac{dy}{dx} = -\frac{Px^2}{2EI} + C_1$ (1)

Integrating → $y = -\frac{Px^3}{6EI} + C_1x + C_2$ (2)

B.C.'s: $\boxed{1}$ $y = 0$ @ $x = L$ $\boxed{2}$ $\frac{dy}{dx} = 0$ @ $x = L$

Substituting *B.C.* $\boxed{2}$ in equation (1) we get:

$$C_1 = \frac{PL^2}{2EI}$$

→ $y = -\frac{Px^3}{6EI} + \frac{PL^2}{2EI}x + C_2$ (3)

Substituting *B.C.* $\boxed{1}$ in equation (3) we get:

$$0 = -\frac{PL^3}{6EI} + \frac{PL^3}{2EI} + C_2$$

→ $C_2 = -\frac{PL^3}{3EI}$

Thus, $\boxed{y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3)}$

and the slope $\boxed{\frac{dy}{dx} = \frac{P}{2EI}(L^2 - x^2)}$

Maximum slope occurs when: $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$ → $x = 0$

$$\boxed{\left(\frac{dy}{dx}\right)_{x=0} = \frac{PL^2}{2EI}}$$
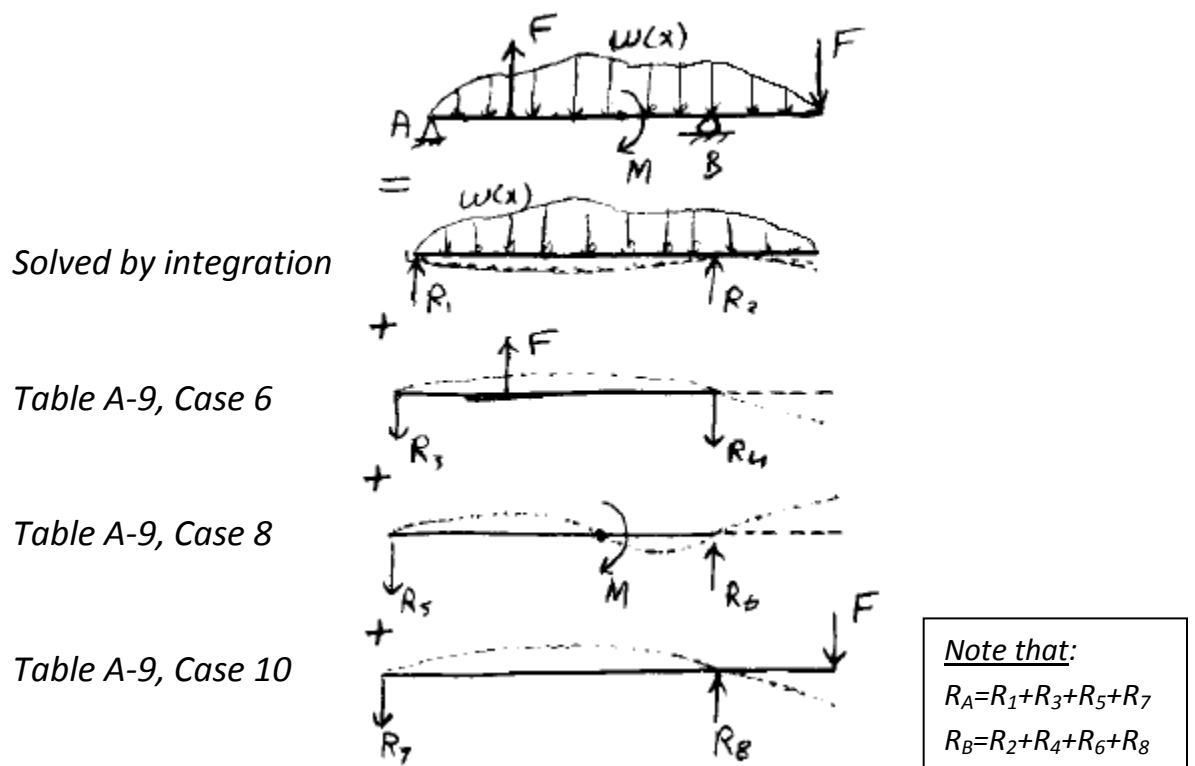
Beam Deflections by Superposition

Superposition resolves the effect of <u>combined loading</u> on a structure by determining the effects of <u>each load separately</u> and adding the results algebraically.

> *Superposition: Resolve, Find, Add.*

> The resulting reactions, deflections and slopes of <u>all common types</u> of loading and boundary conditions are available and can be found in specialized books such as "*Roak's Formulas for Stress and Strain*".

❖ Table <u>A-9</u> gives the results for some common cases.

<u>Note</u>: superposition is <u>valid</u> as long as the <u>deflections are small</u> such that the deformation resulting from one type of loading will not alter the geometry. Also, each effect should be <u>linearly related</u> to the load that produces it.

Example:

Solved by integration

Table A-9, Case 6

Table A-9, Case 8

Table A-9, Case 10

Note that:
$R_A = R_1 + R_3 + R_5 + R_7$
$R_B = R_2 + R_4 + R_6 + R_8$

See **Examples 4-3 & 4-4** from text

Beam Deflections by Singularity Functions

As shown in *Ch. 3*, singularity functions can be used to write an expression for loading over a range of discontinuities.

- The loading intensity function $q(x)$ can be integrated four times to obtain the deflection equation $y(x)$.

(*Recall the singularity functions integration and evaluation rules which has been introduced in Ch.3*)

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*See **Examples 4-5 & 4-6** from text*

</div>

Strain Energy

When loads are applied to an elastic member, the member will elastically deform, thus transferring the work done by the load into potential energy called *strain energy*.

- If a load "F" is applied to a member and as a result the member deformed a distance "y", then the work done by the load is (*if the relation between the load and deformation is linear*):

$$U = \frac{1}{2}Fy \qquad \text{but,} \qquad y = \frac{F}{k} \longleftarrow \text{spring rate}$$

$$\rightarrow \quad \boxed{U = \frac{F^2}{2k}}$$

- ➢ This equation defines the <u>strain energy in general</u> where the load can also mean a torque or moment provided that consistent units are used for "k".
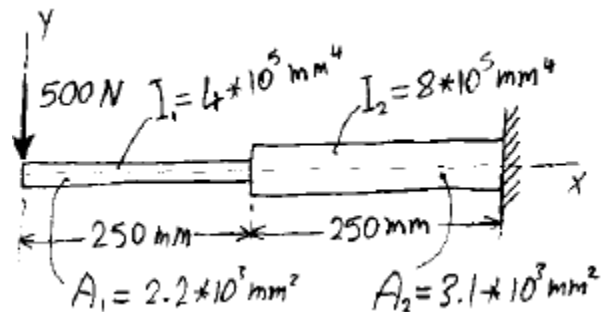
- Therefore, the strain energy for <u>different types of loading</u> can be defined as follows:

Loading Type	Factors Involved	Strain Energy (All factors are constant with x)	Strain Energy (General expression)
Axial Force	F, E, A	$U = \dfrac{F^2 L}{2EA}$	$U = \displaystyle\int_0^L \dfrac{F^2}{2EA} \, dx$
Bending Moment	M, E, I	$U = \dfrac{M^2 L}{2EI}$	$U = \displaystyle\int_0^L \dfrac{M^2}{2EI} \, dx$
Torque	T, G, J	$U = \dfrac{T^2 L}{2GJ}$	$U = \displaystyle\int_0^L \dfrac{T^2}{2GJ} \, dx$
Transverse Shear	V, G, A	$U = C\dfrac{V^2 L}{2GA}$	$U = \displaystyle\int_0^L C\dfrac{V^2}{2GA} \, dx$

- Where C is the cross-section correction factor (Table 4-1 in text) :

$$\begin{cases} C = 1.2 \text{ for rectangular sections} \\ C = 1.11 \text{ for circular sections} \\ C = 2 \text{ for thin-walled circular sections} \\ C = 1 \text{ for box and structural sections (\textit{where only the \underline{area of the}}} \\ \quad\quad \textit{\underline{web} is used)} \end{cases}$$

Example: The cantilever beam shown has a stepped square cross-section and it is made of steel ($E = 210 \ GPa, G = 75 \ GPa$). Find the stain energy in the beam.

$I_1 = 4 \times 10^5 \ mm^4$ $I_2 = 8 \times 10^5 \ mm^4$
500 N
250 mm   250 mm
$A_1 = 2.2 \times 10^3 \ mm^2$ $A_2 = 3.1 \times 10^3 \ mm^2$

Solution:

$$M = 500 \ x \ N.mm \quad \& \quad V = 500 \ N$$

Due to moment $\longrightarrow$ $U_M = U_{M_1} + U_{M_2} = \displaystyle\int_0^{250} \dfrac{M^2}{2EI_1} \, dx + \int_{250}^{500} \dfrac{M^2}{2EI_2} \, dx \quad =$

$$\dfrac{1}{2E}\left[\int_0^{250} \dfrac{25\times 10^4 \, x^2}{4\times 10^5} \, dx + \int_{250}^{500} \dfrac{25\times 10^4 \, x^2}{8\times 10^5} \, dx\right] = \dfrac{1}{2(210\times 10^3)}\left[\left|\dfrac{25 \, x^3}{120}\right|_0^{250} + \left|\dfrac{25 \, x^3}{240}\right|_{250}^{500}\right]$$

➔ $U_M = 34.877 \ mJ$

Due to shear ⟶ $U_V = U_{V_1} + U_{V_2} = C\dfrac{V^2 L_1}{2GA_1} + C\dfrac{V^2 L_2}{2GA_2}$

Very small compared to the strain energy due to moment

$= 1.2\dfrac{500^2}{2(75\times10^3)}\left(\dfrac{250}{2.2\times10^3} + \dfrac{250}{3.1\times10^3}\right) = 0.389 \ mJ$

➔ $\boxed{U_{Total} = U_M + U_V = 35.266 \ mJ}$

Usually the strain energy due to shear is neglected for long beams

Castigliano's Theorem

Castigliano's theorem is one of the energy methods (*based on strain energy*) and it can be used for solving a wide range of deflection problems.

- Castigliano's theorem states that when a body is elastically deformed by a system of loads, the deflection at any point "*P*" in any direction "*a*" is equal to the partial derivative of the strain energy with respect to a load at "*P*" in the direction "*a*".

The theory applies to both linear and rotational deflections

$$\boxed{\delta_i = \frac{\partial U}{\partial F_i}}$$

Where δ_i is the displacement of the point of application of the force F_i in the direction of F_i

or, $$\boxed{\theta_i = \frac{\partial U}{\partial M_i}}$$

Where θ_i is the rotation (*in radians*) at the point of application of the moment M_i in the direction of M_i

- It should be clear that Castiglione's theorem finds the deflection at the point of application of the load in the direction of the load.

Q: What about if there is no load at the desired point in the desired direction?

A: We apply a fictitious *"dummy"* load "*Q*" at that point in the desired direction.

Procedure for using Castigliano's theorem:

- Obtain an expression for the total strain due to all loads acting on the member (*including the dummy load if one is used*).
- Find the deflection of the desired point (*in the desired direction*) as:

$$\delta_i = \frac{\partial U}{\partial F_i}$$

If the deflection turns out to be negative, it means that it is in the opposite direction of the load

or if a dummy load Q_i is used:

$$\delta_i = \left[\frac{\partial U}{\partial Q_i}\right]_{Q_i=0}$$

After taking the derivative, set $Q_i = 0$ to find the deflection

Note: If the strain energy is in the form of integral, it is better to take the partial derivative with respect to the load "F_i" before integrating.

*See **Examples 4-8 & 4-9** from text*

Example: For the cantilevered bar subjected to the horizontal load "*P*" as shown. Find the vertical deflection at the free end "*Point A*". (*Neglect the transverse shear*).

Solution:

The components that define the total strain energy are:

- Bending in *AB*, where $M_{AB} = P\,y$
- Bending in *BC*, where $M_{BC} = P\,h + Q\,x$
- Axial load in $AB = Q$
- Axial load in $BC = P$
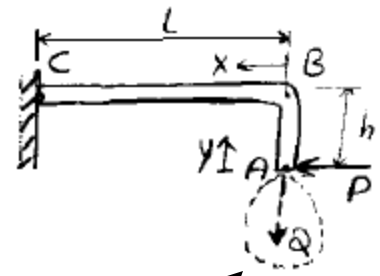
Since there is no vertical load acting at point "A", we add a dummy load Q

Thus, the total strain energy is:

Since the axial loads are of constant magnitude

$$U = \int_0^h \frac{P^2 y^2}{2EI}\,dy + \int_0^L \frac{(Ph + Qx)^2}{2EI}\,dx + \frac{Q^2 h}{2EA} + \frac{P^2 L}{2EA}$$

The vertical deflection at "A" is:

$$\delta_A = \left[\frac{\partial U}{\partial Q}\right]_{Q=0} = \int_0^L \frac{x(Ph + Qx)}{EI}dx + \frac{Qh}{EA} = \frac{Phl^2}{2EI} + \frac{Ql^3}{3EI} + \frac{Qh}{EA}$$

Set $Q = 0$

➜ $$\boxed{\delta_A = \frac{Phl^2}{2EI}}$$

The expressions for finding deflections using Castigliano's theorem can be simplified and written as:

$$\delta_i = \frac{\partial U}{\partial F_i} = \boxed{\int \frac{1}{AE}\left(F\frac{\partial F}{\partial F_i}\right)dx} \qquad for\ Tension\ or\ Compression$$

$$\delta_i = \frac{\partial U}{\partial F_i} = \boxed{\int \frac{1}{EI}\left(M\frac{\partial M}{\partial F_i}\right)dx} \qquad for\ Bending$$

$$\theta_i = \frac{\partial U}{\partial T_i} = \boxed{\int \frac{1}{GJ}\left(T\frac{\partial T}{\partial T_i}\right)dx} \qquad for\ Torsion$$

> See **Example 4-11** from text

Deflection of Curved Beams

Examples of curved beams include machine frames, springs, clips, etc. The deflection of curved beams can be found using Castigliano's theorem.

- Consider a curved beam subjected to "*in-Plane*" loading as shown:
    - Note that the load "*F*" is purely shear at $\theta = 0$, and purely normal at $\theta = 90°$. Thus, polar coordinates are used to represent the internal loading.
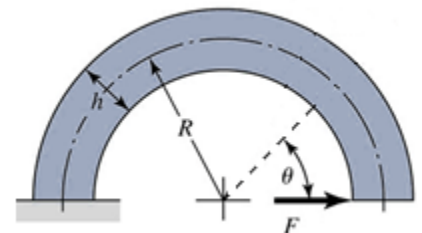
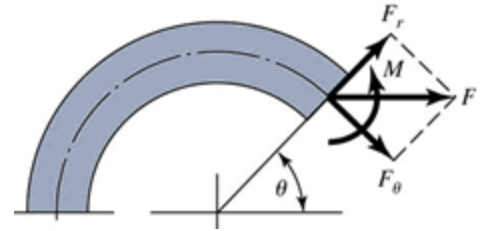- The internal loading acting at any angle θ is:

    Shear: $\qquad\qquad\qquad F_r = F \cos \theta$

    Axial: $\qquad\qquad\qquad F_\theta = F \sin \theta$

    Bending Moment: $\quad M = F R \sin \theta$

- The total strain energy in the beam consists of four components:
    - Strain energy due to <u>bending moment</u> M:

    $$U_1 = \int \frac{M^2}{2AeE} \, d\theta$$

    Where "e" is the eccentricity $(e = R - r_n)$

    However, when the radius "R" is <u>large compared to</u> the height "h" the strain energy due to moment can be <u>approximated</u> as:

    $$U_1 = \int \frac{M^2 R}{2EI} \, d\theta \qquad for \frac{R}{h} > 10$$

    - The strain energy due to <u>normal "axial" force</u> F_θ has <u>two components</u>:
        - Due to the <u>axial force</u> F_θ:
        $$U_2 = \int \frac{F_\theta^2 R}{2AE} \, d\theta$$

        - Due to the <u>moment produced by the axial force</u> F_θ (which has an opposite direction to the moment M):

        $$U_3 = - \int \frac{MF_\theta}{AE} \, d\theta$$

        Because it causes deflection in the direction opposite to the force" F"

    - The strain energy due to <u>shear force</u> F_r

    $$U_4 = \int C \frac{F_r^2 R}{2AG} \, d\theta$$

    Where "C" is the cross-section correction factor

- According to Castigliano's theorem, the deflection in the direction of the force "F" is found as:

$$\delta = \frac{\partial U}{\partial F} = \frac{\partial U_1}{\partial F} + \frac{\partial U_2}{\partial F} + \frac{\partial U_3}{\partial F} + \frac{\partial U_4}{\partial F}$$

Taking the partial derivatives and integrating between 0 to π, then simplifying we get:

$$\delta = \frac{\pi F R^2}{2AeE} - \frac{\pi F R}{2AE} + \frac{\pi C F R}{2AG}$$

➤ Note that the <u>first term</u> (*which is due to moment*) will be <u>much larger</u> than the other two terms when R is large, since it has R^2, and thus the other terms <u>can be neglected</u> and we can use the approximate expression of strain energy for $R/h > 10$ which gives an approximate result:

$$\delta \cong \frac{\pi F R^2}{2EI}$$

Example: For the beam shown, find the <u>vertical</u> deflection at point "A" considering the moment only and knowing that $R/h{>}10$ for the curved portion.
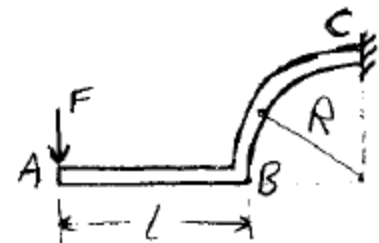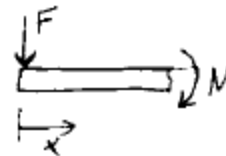

<u>Solution</u>:

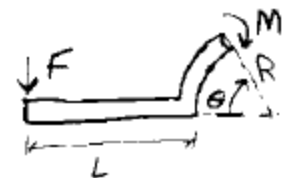Since the moment only is considered, the total strain energy has two components:

Due to moment in AB :

$$M_{AB} = Fx$$

Due to moment in BC :

$$M_{BC} = FL + FR\,(\,1 - \cos\theta\,)$$

$$U = U_{AB} + U_{BC} = \int_0^L \frac{M_{AB}^2}{2EI}\,dx + \int_0^{\pi/2} \frac{M_{BC}^2}{2EI}\,R\,d\theta$$

$$\delta = \frac{\partial U}{\partial F} = \int_0^L (Fx)\frac{x}{EI}dx + \int_0^{\pi/2} (FL + FR(1 - \cos\theta))\frac{(L + R(1 - cos\theta))}{EI}Rd\theta$$

➔ $$\boxed{\delta = \frac{F}{12EI}\{4L^3 + 3R[2\pi L^2 + 4(\pi - 2)LR + (3\pi - 8)R^2]\}}$$

Statically Indeterminate Problems

A statically indeterminate problem has <u>more unknown reactions</u> than what can be found using static equilibrium equations (*i.e., number of unknown is more than the number of available equations*).

- The additional supports that are *not necessary to keep the member in equilibrium* are called <u>redundant supports</u> and their reactions are called "*redundant reactions*".
- The number of redundant reactions is defined as the <u>degree of indeterminacy</u>, and the same number of additional equations is needed to solve the problem.

How to solve statically indeterminate problems?

- Using <u>integration or singularity functions</u>:
    1. Write the equations of static equilibrium in terms of applied loads and the unknown reactions.
    2. Obtain the expressions for slope and deflection of the beam.
    3. Apply the boundary conditions consistent with the restraints "reactions"
    4. Solve the equations obtained from steps *1* and *3*.

- Using <u>Superposition</u>:
    - Choose the redundant reaction(s).
    - Write the equations of static equilibrium for the remaining reactions in terms of applied loads and the redundant reaction.

- Using superposition divide the loading where once the applied loading is present and the redundant reaction is removed, and the other time the applied load is removed and the redundant reaction is present.
- Find the deflection (*can be found from tables or using Castigliano's theorem*) at the location of the redundant reaction in both cases and use that as an additional compatibility equation where the sum of the deflections is zero. (*note that if the redundant reaction is a moment, the corresponding deflection will be slope*)
- Solve the compatibility equation(s) for the redundant reaction, and then find the remaining reactions from the equilibrium equations.

Example:

Choosing R_2 & M_2 as the two redundant reactions
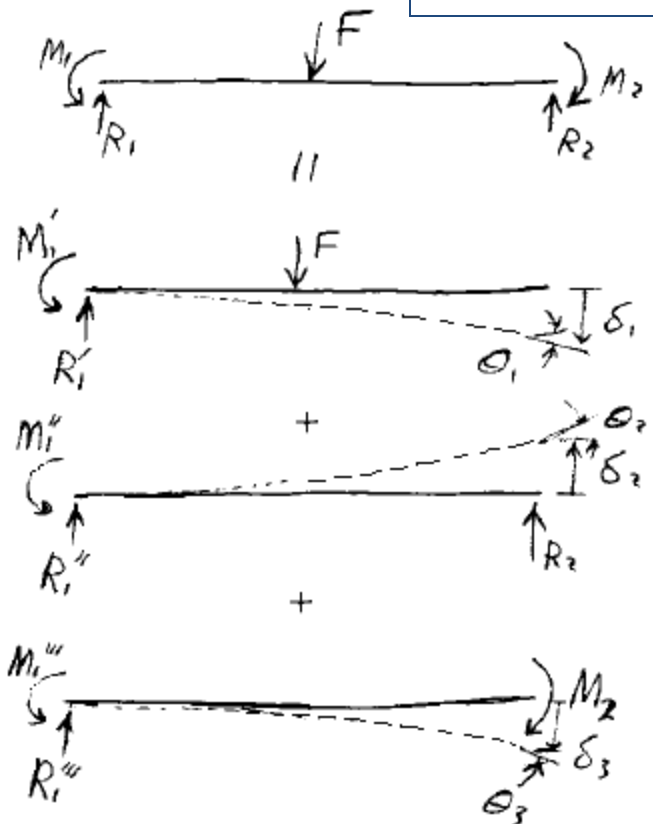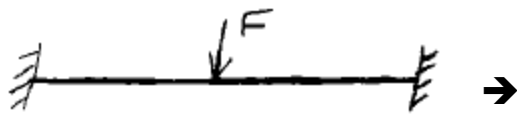
Compatibility equations:
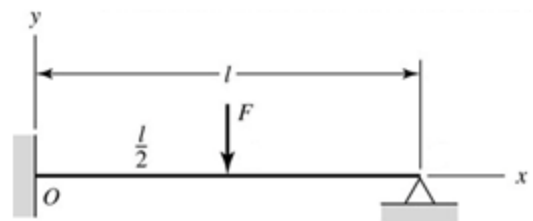
Solved for the redundant reactions R_2 & M_2

$$\delta_1 + \delta_2 + \delta_3 = 0$$
$$\theta_1 + \theta_2 + \theta_3 = 0$$
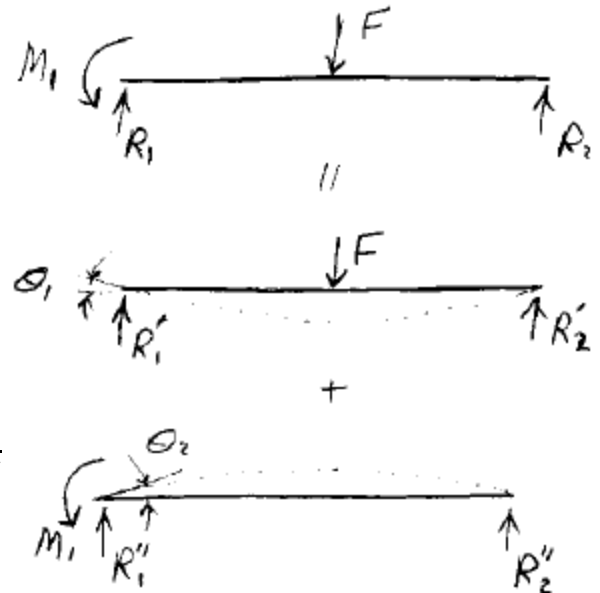
where:

Can be used to find R_1 & M_1

$$M_1 = M_1{}' + M_1{}'' + M_1{}'''$$
$$R_1 = R_1{}' + R_1{}'' + R_1{}'''$$

Example: Determine the reactions for the beam shown using superposition.

Solution: _Choosing the reaction M_1 to be the redundant reaction_ ➘

From equilibrium we get:

$$R_1 = \frac{F}{2} + \frac{M_1}{L} \qquad (1)$$

$$R_2 = \frac{F}{2} - \frac{M_1}{L} \qquad (2)$$

From Table A-9 (_case 5_):

$$\theta_1 = \left[\frac{dy}{dx}\right]_{x=0} = \left[\frac{F}{48EI}(12x^2 - 3L^2)\right]_{x=0} = -\frac{FL^2}{16EI}$$

From Table A-9 (_case 8_) and setting _a = 0_ we get:

$$\theta_2 = \left[\frac{dy}{dx}\right]_{x=0} = \left[\frac{M_1}{6EIL}(3x^2 + 2L^2)\right]_{x=0} = \frac{M_1 L}{3EI}$$

Compatibility:

$$\theta_1 + \theta_2 = 0 \qquad ➔ \qquad \frac{M_1 L}{3EI} - \frac{FL^2}{16EI} = 0 \qquad ➔ \qquad \boxed{M_1 = \frac{3FL}{16}}$$

Substituting in equations 1 & 2 we get:

$$➔ \qquad \boxed{R_1 = \frac{11F}{16}} \qquad \& \qquad \boxed{R_2 = \frac{5F}{16}}$$

See the solution in **Example 4-14** in the text book

Compression Members

The analysis and design of compression members "Columns" is different from members loaded in tension.

A column is a straight and long (_relative to its cross-section_) bar that is subjected to a compressive axial load. A column can fail due to buckling (_a sudden, large lateral deflection_) before the compressive stress in the column reach its allowable (_yield_) value.

- Buckling or "*elastic instability*" can cause <u>catastrophic failure</u> of structures. To understand why buckling happens, it is important to understand equilibrium regimes. There are <u>three</u> states of equilibrium:
  - *Stable Equilibrium*: when the member is moved it tends to return to its original equilibrium position.

  - *Neutral Equilibrium*: when the member is moved from its equilibrium position, it is still in equilibrium at the displaced position.
  - *Unstable Equilibrium*: when the member is moved from its equilibrium position it becomes imbalanced and accelerates away from its equilibrium position.
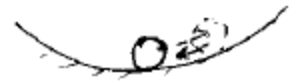
- With <u>increasing compressive loading</u>, the state of equilibrium of a column changes from <u>stable to neutral to unstable</u>.

- The load that causes a column to become <u>unstable</u> is called the "*critical buckling load*" where under unstable equilibrium condition; any small lateral movement will cause buckling (catastrophic failure).

- According to <u>geometry and loading</u>, columns can be categorized as:
  - Long columns with central loading.
  - Intermediate length columns with central loading.
  - Columns with eccentric loading.
  - Stratus or short columns with eccentric loading.

Long Columns with Central Loading

The critical "buckling" load for long columns with central loading is predicted by *Euler* formula and it depends on:
- <u>End conditions</u>: four types of end conditions; *pinned-pinned*, *fixed-fixed*, *fixed-pinned* and *fixed-free*.
- The column <u>material</u> "E".
- <u>Geometry</u>: length and cross-section.

➤ Consider a pinned-pinned column of length "L" and subjected to axial load "P".
  - Assume that the column became slightly bent.
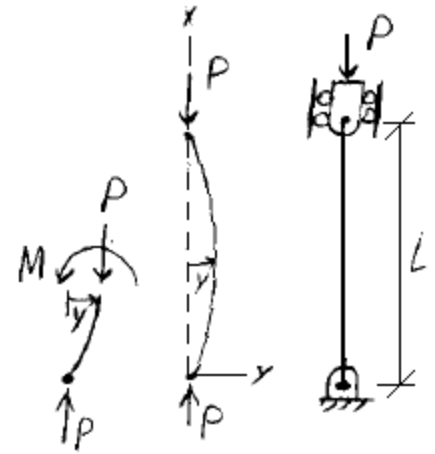
The bending moment developed in the column is:

$$M = -Py$$

Using the beam governing equation:

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

➔ $$\frac{d^2y}{dx^2} = -\frac{P}{EI}y$$

or

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

Which is a homogeneous, second order differential equation. The <u>general solution</u> for the differential equation is:

$$y = A \sin\sqrt{\frac{P}{EI}}x + B \cos\sqrt{\frac{P}{EI}}x \qquad (1)$$

The equation can be evaluated using the boundary conditions:
1. $y = 0 @ x = 0$
2. $y = 0 @ x = L$

From *B.C.1*: ➔ $B = 0$

From *B.C.2*: ➔ $0 = A \sin\sqrt{\frac{P}{EI}}L$

The constant "A" cannot be zero because that is a trivia solution which means that the column will not be buckle. Thus,

$$\sin\sqrt{\frac{P}{EI}}L = 0$$

This is satisfied when:

$$\sqrt{\frac{P}{EI}}L = n\pi \qquad\qquad \text{where } n = 1,2,3 \dots$$

Solving for "P" we get:

$$P = \frac{n^2 \pi^2 E I}{L^2}$$

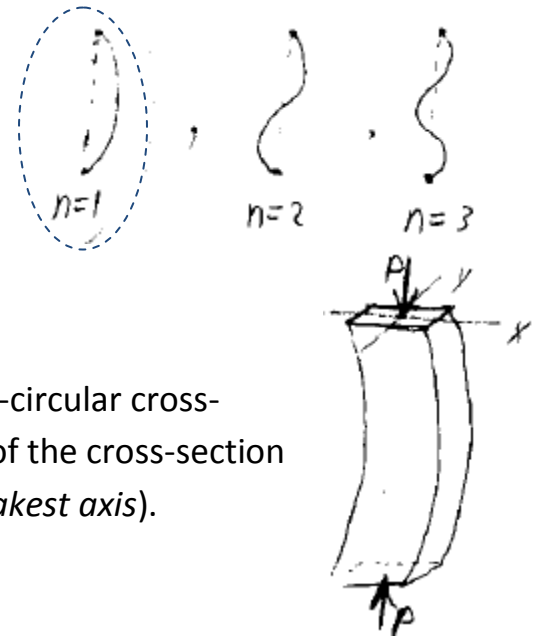where the smallest "P" is obtained for $n = 1$ (*The first mode of buckling*)

Thus, the critical load "P_{cr}" for a column with pinned ends is:

➜ $P_{cr} = \dfrac{\pi^2 E I}{L^2}$ *Euler formula*

Substituting back in equation (*1*) we get the deflection curve:

$$y = A \sin\frac{\pi x}{L}$$

which is a half-wave

➢ It should be realized that for columns with non-circular cross-section, the column will buckle about the axis of the cross-section having the smallest moment of inertia (*the weakest axis*).

- Using the relation $I = A k^2$ where "A" is the cross-sectional area and "k" is the radius of gyration, we can write *Euler* formula in a more convenient form:

Critical buckling stress ➜ $\sigma_{cr} = \dfrac{P_{cr}}{A} = \dfrac{\pi^2 E}{(L/k)^2}$

> The stress value that will cause a column to be in unstable equilibrium condition

where the ratio (L/k) is called the "*slenderness ratio*"

➢ This ratio (rather than the actual length) is used to classify columns in to length categories.

- The critical loads for columns with different end conditions can be obtained by solving the differential equation and it can be seen by comparing the buckled shapes.

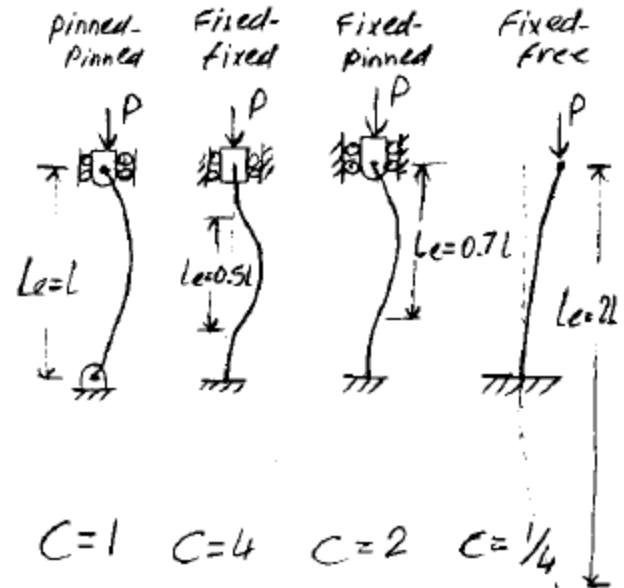- *Euler* formula for <u>different end conditions</u> can be written in the form:

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

where L_e is the "effective length"

Or alternatively *Euler* formula can be written as:

➔ $$\boxed{P_{cr} = \frac{C\pi^2 EI}{L^2}}$$ or $$\boxed{\sigma_{cr} = \frac{P_{cr}}{A} = \frac{C\pi^2 E}{(L/k)^2}}$$

where C is the "end-condition constant"

❖ In reality it is not possible to fix an end (*even if it is welded*); thus, recommended "<u>realistic</u>" values of C are given in *Table 4-2*.

- Plotting the *Euler* <u>critical stress vs. the slenderness ratio</u> we get:

  ➢ It is noted that for slenderness ratio less than that of point "T"; buckling occurs at a <u>stress value less than</u> that predicted by *Euler* formula (*it follows the dashed line*).
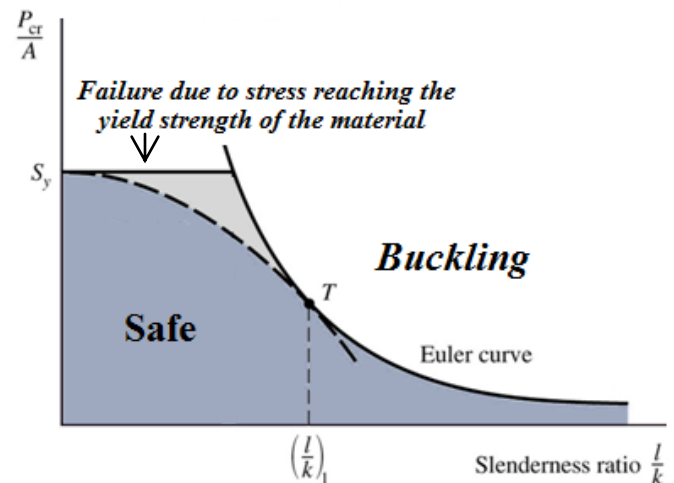
- Thus, the *Euler* formula is used for <u>slenderness ratios larger than</u> $(L/k)_1$ which is given as:

Limiting slenderness ratio ➔ $$\boxed{\left(\frac{L}{k}\right)_1 = \sqrt{\frac{2\pi^2 CE}{S_y}}}$$ Long columns: $\left(\frac{L}{k}\right) > \left(\frac{L}{k}\right)_1$

Intermediate-Length Columns with Central Loading

For columns with slenderness ratio <u>smaller than</u> $(L/k)_1$, *Euler* formula cannot predict the critical buckling load. For this range the *Johnson formula* is used which is a parabolic fit between "S_y" and point "T".

- The *Johnson* formula predicts the critical stress as:

$$\boxed{\sigma_{cr} = \frac{P_{cr}}{A} = S_y - \frac{S_y^2}{4\pi^2 CE}\left(\frac{L}{k}\right)^2} \quad for \quad \frac{L}{k} < \left(\frac{L}{k}\right)_1$$

Or alternatively using the effective length "L_e" it is written as:

$$\sigma_{cr} = S_y - \frac{S_y^2}{4\pi^2 E}\left(\frac{L_e}{k}\right)^2$$

Example: A column with one end fixed and the other free is to be made of *Aluminum alloy 2014 (E=72GPa, S_y=97MPa).* The column cross-sectional area is to be 600 *mm²* and its length is 2.5 *m.*

Find the critical buckling load for the following cross-sections:

    a) A solid round bar
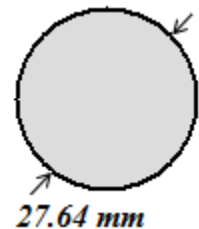   *b)* A solid square bar

<u>Solution</u>:

Fixed-Free: $C = 1/4$

a) Round bar

$$A = \frac{\pi}{4}d^2 \quad \rightarrow \quad d = \sqrt{\frac{4A}{\pi}} = \sqrt{\frac{4(600)}{\pi}} = 27.64 \ mm$$

27.64 mm

Moment of Inertia: $I = \frac{\pi}{64}d^4 = \frac{\pi}{64}(27.64)^4 = 28650 \ mm^4$

Radius of gyration: $k = \sqrt{\frac{I}{A}} = \sqrt{\frac{28650}{600}} = 6.91 \ mm$

Slenderness ratio: $\left(\frac{L}{k}\right) = \frac{2500}{6.91} = 361.8$

Limiting slenderness ratio:

$$\left(\frac{L}{k}\right)_1 = \sqrt{\frac{2\pi^2 CE}{S_y}} = \sqrt{\frac{2\pi^2(0.25)(72 \times 10^9)}{97 \times 10^6}} = 60.52$$

Comparing: $361.8 > 60.52$ ➔   Long column (*use Euler formula*)

$$\text{➔} \quad P_{cr} = \frac{C\pi^2 EI}{L^2} = \frac{(0.25)\pi^2(72 \times 10^9)(28.65 \times 10^{-9})}{2.5^2} = \boxed{814.4 \ N}$$

b) Square bar

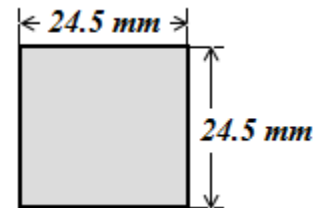$$A = b^2 \quad \text{➔} \quad b = \sqrt{A} = \sqrt{600} = 24.5 \ mm$$

$$I = \frac{b^4}{12} = \frac{24.5^4}{12} = 30000 \ mm^4$$

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{30000}{600}} = 7.07 \ mm$$

$$\left(\frac{L}{k}\right) = \frac{2500}{7.07} = 353.6$$

$353.6 > 60.52$ ➔   Long column (*use Euler formula*)

$$\text{➔} \quad P_{cr} = \frac{(0.25)\pi^2(72 \times 10^9)(30 \times 10^{-9})}{2.5^2} = \boxed{852.7 \ N}$$

➢ The example shows that for the same cross-sectional area the square section has <u>slightly higher</u> critical buckling load than the circular one.

➢ To increase the critical buckling load without increasing the cross-sectional area, <u>hollow tube sections</u> are used (*since they have higher moment of inertia*).

➢ When <u>thin-walled</u> hollow tubes are used, <u>wall buckling</u> is considered too. For this reason <u>circular tubes are better than square tubes</u> because the buckling load of curved walls is higher than that of flat walls.

> *The critical stress causing <u>wall buckling of thin-wall circular tubes</u> can be found as:*
> $$\sigma_{cr} = \frac{Et}{R\sqrt{3(1-v^2)}}$$
> *E & v are young's modulus and poisson's ratio and R & t are the tube radius and wall thickness.*

Columns with Eccentric Loading

In most applications the load is not applied at the centroid of the cross-section, but rather it is eccentric.
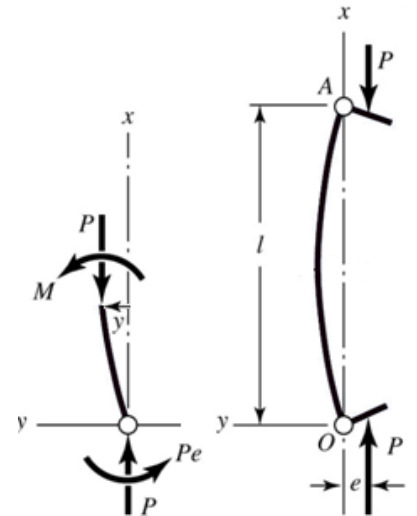
- The distance between the centroidal axis and the point of load application is called the eccentricity "e".

- Considering a pinned-pinned column with eccentric load "P".

  The bending moment developed in the column is:

  $$M = -P(e + y)$$

  Using the beam governing equation $\dfrac{d^2y}{dx^2} = \dfrac{M}{EI}$ we get:

  $$\frac{d^2y}{dx^2} + \frac{P}{EI}y = -\frac{Pe}{EI}$$

  Using the same procedure used before to solve the differential equation, and the same *B.C.s* we can obtain the deflection equation. The maximum deflection occurs at mid-span ($x=L/2$) and it is found to be:

  $$\delta = y_{max} = e\left[sec\left(\frac{L}{2}\sqrt{\frac{P}{EI}}\right) - 1\right]$$

  and thus, maximum moment at mid-span is:

  $$M_{max} = P(e + \delta) = Pe\,sec\left(\frac{L}{2}\sqrt{\frac{P}{EI}}\right)$$

  The maximum compressive stress at mid-span has <u>two components</u>; axial and bending:

  $$\sigma_c = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{Mc}{Ak^2}$$

  > *where "c" is the distance from the neutral axis to the outer surface*

Substituting the value of the maximum moment at mid-span we get:

Max compressive stress at mid span

$$\sigma_c = \frac{P}{A}\left[1 + \frac{ec}{k^2} \, sec\left(\frac{L}{2k}\sqrt{\frac{P}{EA}}\right)\right]$$

where the term (ec/k^2) is called the <u>*eccentricity ratio*</u>

- If we set the maximum value of compressive strength to be <u>equal to the compressive yield strength</u> of the column material "S_{yc}" we can write:

➔ $\boxed{P} = \dfrac{A\,S_{yc}}{1 + \left(\frac{ec}{k^2}\right)\,sec\left(\frac{L_e}{2k}\sqrt{\frac{\boxed{P}}{EA}}\right)}$ *The secant equation*

Where L_e is the effective length for any end condition

- Note that "**P**" appears twice in the equation, thus this equation needs an <u>iterative process</u> to find the critical buckling load.
  - ➢ For the <u>first iteration</u> the critical buckling load obtained from *Euler* formula "P_{cr}" is used, and the iterations continue until both sides converge.

Critical load decreases with increasing eccentricity ratio
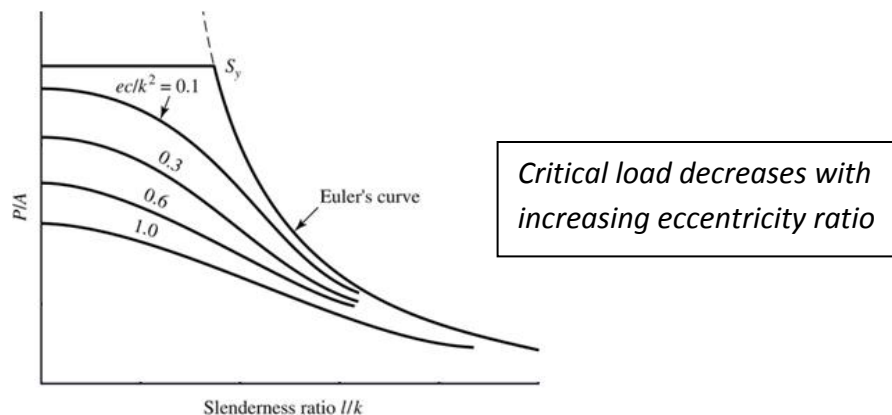
Struts, or Short Compression Members

When a short bar is loaded in compression by a load "*P*" acting along the centroidal axis, the stress increases uniformly as $\sigma = P/A$ until it reaches the yield strength of the material.

However, if the load is eccentric an additional component of stress, due to the bending moment, is introduced:

$$\sigma_c = \frac{P}{A} + \frac{My}{I} = \frac{P}{A} + \frac{(Pe)y}{I} = \frac{P}{A}\left(1 + \frac{ey}{k^2}\right)$$

- The maximum compressive stress occurs at point "B" on the surface where $y = c$

$$\rightarrow \quad \boxed{\sigma_c = \frac{P}{A}\left(1 + \frac{ec}{k^2}\right)}$$
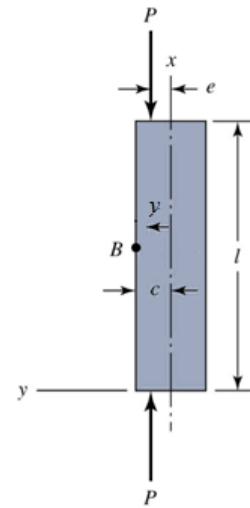
➤ If the maximum stress value is S_y, then we can solve for the critical load P_{cr}.

- To distinguish short columns (*or struts*) from long columns with eccentric loading the slenderness ratio is used, where the limiting value is:

$$\left(\frac{L}{k}\right)_2 = 0.282\sqrt{\frac{AE}{P}}$$

Limiting value of slenderness ratio for columns with eccentric loading

$$\text{if } \frac{L}{k} > \left(\frac{L}{k}\right)_2 \quad \rightarrow \quad Long\ column\ (use\ secant\ formula)$$

$$\text{if } \frac{L}{k} < \left(\frac{L}{k}\right)_2 \quad \rightarrow \quad Strut\ (short\ column)$$

See **Example 4-20** from text